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On Certain Identities in the Theory of Matrices.

By Henry Taber, *Clark University.*

1. In this paper I consider some applications to the general theory of matrices of conceptions familiar in quaternions (the separation into scalar and vector parts, the conjugate, etc.) which I have extended in a previous paper in this Journal, Vol. XII, to matrices of order higher than the second. By this means I find that the identical equation, the identical relations between two matrices (constituting with the identical equation the catena of equations), and Sylvester's formula, may be exhibited as explicit identities, and that the law of latency is an immediate corollary of an explicitly identical proposition. Moreover, by this means I find expressions for the coefficients of the equations of the catena, as simple functions of the sum of the latent roots of the powers and products of powers of the matrices involved. I also prove the extension of the conceptions met with in quaternions without regarding the matrix as an operator linear in and distributive over the units of an algebra, as in establishing these conceptions in the paper above referred to: thus it will be shown that a matrix of the third or of higher order, like a matrix of order two (quaternion), is separable into a scalar and a non-scalar (or vector) part, and that the vector part of a matrix of order ω is further separable into $\omega - 1$ sub-vector parts, which may be termed the first, second, etc., and $(\omega - 1)^{\text{th}}$ vector parts. The separation of a matrix into a scalar and ω vector parts is of much importance in the applications considered in this paper.* In regard to the extension of the conception of the conjugate,† it will be shown that a matrix of order ω has $\omega - 1$ conjugates which

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* It will appear later that the identical equation is merely a corollary (immediate) of this separation of a matrix into its ω parts.

† The term *conjugate* is employed in quaternions by Hamilton and Tait with two different significations: 1, to denote a certain function of a quaternion; and 2, to denote a different function, the *converse* (Peirce) or *transverse* (Cayley and Sylvester) of a linear vector operator (matrix of the second or of the third order). (The converse of a given matrix is the matrix obtained by interchanging its rows and

reduce, when $\omega = 2$, to the single conjugate of quaternions; it will appear that the product of a matrix of order ω and its $\omega - 1$ conjugates is a scalar, equal to the content of the matrix; and, following the analogy of quaternions, I define the tensor of the matrix as the ω^{th} root of this product, which is commutative. Further, I shall show that a matrix of order ω may be represented as the product of its tensor into $\omega - 1$ versors, each versor being dependent upon a single parameter contained in all its ω terms, or into a single versor dependent upon $\omega - 1$ parameters, each of which enters into its ω terms. The functions of the parameters which appear in the versors of a matrix are simple extensions of the trigonometrical functions, to which (or to the hyperbolic functions) they reduce when $\omega = 2$.

I shall prove these propositions for matrices of the third order, but the method employed is perfectly general.

2. *A matrix n of the third order, whose latent roots g_1, g_2, g_3 are all distinct, may be represented as follows:*

$$n = \varpi \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} \varpi^{-1},$$

*where ϖ is not vacuous.** Let

$$n = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \varpi = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad \varpi^{-1} = \begin{pmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{pmatrix},$$

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columns.) A quaternion may be regarded as a matrix of the second order, and its conjugate (in Hamilton's and Tait's sense) as a matrix or linear vector operator, is not identical with its conjugate as a quaternion; so it is best to restrict the term in quaternions to its first signification. As a term is needed to express the same function of a matrix of order higher than the second, which the conjugate (with the first signification) is of a quaternion, or matrix of the second order, I employ the term conjugate for this purpose.

* From this form of n can obviously be obtained immediately demonstrations of the identical equation, of Sylvester's formula, and of the law of latency. It also gives an expression, more simple than Sylvester's formula, for any function of a matrix whose latent roots are all distinct, provided the function can be expressed in terms of positive integral powers of n, which is not always the case. For then we have

$$Fn = \varpi \begin{pmatrix} Fg_1 & 0 & 0 \\ 0 & Fg_2 & 0 \\ 0 & 0 & Fg_3 \end{pmatrix} \varpi^{-1}.$$

where B_{11}, B_{12}, etc., denote the first minors of ϖ with respect to b_{11}, b_{12}, etc., respectively, divided by the content of ϖ. If we put

$$\left\{\begin{array}{llll} (a_{11}-g_1)\,b_{11} + a_{12}b_{21} & & + a_{13}b_{31} & = 0, \\ \qquad a_{21}b_{11} + (a_{22}-g_1)\,b_{21} & & + a_{23}b_{31} & = 0, \\ \qquad a_{31}b_{11} + a_{32}b_{21} & & + (a_{33}-g_1)\,b_{31} & = 0, \end{array}\right.$$

$$\left\{\begin{array}{llll} (a_{11}-g_2)\,b_{12} + a_{12}b_{22} & & + a_{13}b_{32} & = 0, \\ \qquad a_{21}b_{12} + (a_{22}-g_2)\,b_{22} & & + a_{23}b_{32} & = 0, \\ \qquad a_{31}b_{12} + a_{32}b_{22} & & + (a_{33}-g_2)\,b_{32} & = 0, \end{array}\right.$$

$$\left\{\begin{array}{llll} (a_{11}-g_3)\,b_{13} + a_{12}b_{23} & & + a_{13}b_{33} & = 0, \\ \qquad a_{21}b_{13} + (a_{22}-g_3)\,b_{23} & & + a_{23}b_{33} & = 0, \\ \qquad a_{31}b_{13} + a_{32}b_{23} & & + (a_{33}-g_3)\,b_{33} & = 0, \end{array}\right.$$

the ratio of the constituents of each column of ϖ will be completely determined. From the group of three equations consisting of the first equation of each of these three sets, each transformed by putting the term containing g in the right-hand member, we obtain immediately

$$\begin{aligned} a_{11} &= g_1b_{11}B_{11} + g_2b_{12}B_{12} + g_3b_{13}B_{13}, \\ a_{12} &= g_1b_{11}B_{21} + g_2b_{12}B_{22} + g_3b_{13}B_{23}, \\ a_{13} &= g_1b_{11}B_{31} + g_2b_{12}B_{32} + g_3b_{13}B_{33}, \end{aligned}$$

three equations that, together with the six equations obtained in like manner from the second equations of each of the three sets, and from the third equations of each of the three sets, constitute the conditions necessary and sufficient that

$$n = \varpi \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} \varpi^{-1}.\text{*}$$

If two of the latent roots are equal, as $g_1 = g_2$, then n may be represented, in the same way, provided the nullity of $n - g_1$ is two. In general, a matrix of order ω is representable in like manner if its latent roots are all distinct, or if any latent root occurs m times, provided the nullity of the matrix, less that latent root, is m.†

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[*] Compare this Journal, Vol. XII, p. 359.

[†] This is readily seen by means of the above scheme of equations determining the constituents of ϖ. It may also be readily proved by regarding a matrix as an operator linear and distributive over the units of an algebra (i. e., as a linear unit, or vector operator). Thus, if of the three latent roots of n, $g_1 = g_2$, then n has three linearly independent axes (the only case in which the matrix is representable as above), only if $n - g_1$ annuls two linearly independent vectors; but then $|\, n - g_1 \,|$ (the determinant of $n - g_1$) has nullity two. See this Journal, Vol. XII, p. 363.

3. The *first conjugate* of a matrix is obtained from the matrix, when in the form of (2), by a cyclic interchange of its latent roots; the *second conjugate*, by a repetition of this cyclic interchange, etc. Thus the first and second conjugates of the nonion n, denoted respectively by $K_1 n$ and $K_2 n$, are

$$K_1 n = \varpi \begin{pmatrix} g_3 & 0 & 0 \\ 0 & g_1 & 0 \\ 0 & 0 & g_2 \end{pmatrix} \varpi^{-1} \quad K_2 n = \varpi \begin{pmatrix} g_2 & 0 & 0 \\ 0 & g_3 & 0 \\ 0 & 0 & g_1 \end{pmatrix} \varpi^{-1}.$$

Obviously $K_2 n = K_1 (K_1 n)$, consequently we may dispense with the subscripts, and write K for K_1, and K^2 for K_2. We have in the case of matrices of the third order, $K.K^2 = K^2.K = K^3 = 1$, whereas in quaternions $K^2 = 1$.

It is very easy to see that the conjugate as defined here is identical, when $\omega = 2$, with the quaternion conjugate. For by (2) any quaternion whose latent roots g_1 and g_2 are distinct, may be represented as

$$q = \varpi \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \varpi^{-1},$$

where ϖ is now a matrix of the second order:

$$\therefore\ Kq = \varpi \begin{pmatrix} g_2 & 0 \\ 0 & g_1 \end{pmatrix} \varpi^{-1} = \varpi \left(- \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} + 2(g_1 + g_2) \right) \varpi^{-1}$$
$$= -q + 2Sq = Sq - Vq,$$

which is the ordinary definition of the quaternion conjugate.

Evidently, in general for matrices of any order, as in quaternions,

$$K(Fn) = F(Kn).$$

4. If we put

$$\varepsilon_1 = \varpi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \varpi^{-1}, \quad \varepsilon_2 = \varpi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \varpi^{-1}, \quad \varepsilon_3 = \varpi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \varpi^{-1},$$

then

$$n = g_1\varepsilon_1 + g_2\varepsilon_2 + g_3\varepsilon_3,$$
$$Kn = g_3\varepsilon_1 + g_1\varepsilon_2 + g_2\varepsilon_3,$$
$$K^2 n = g_2\varepsilon_1 + g_3\varepsilon_2 + g_1\varepsilon_1,$$

and since the ε's are idempotent and mutually nilfactorial, it follows that any symmetric function of the three conjugate matrices n, Kn, $K^2 n$, (involving no other matrix but unity) is the same symmetric function of their latent roots into $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$.

From this theorem can immediately be derived the identical equation and Sylvester's formula as explicit identities, and the law of latency may be shown to be involved in an explicit identity:

A. For by the proposition just proved,

$$\begin{aligned} & n^3 - (g_1 + g_2 + g_3)\, n^2 + (g_2 g_3 + g_3 g_1 + g_1 g_2)\, n - g_1 g_2 g_3 \\ \equiv\; & n^3 - (n + Kn + K^2 n)\, n^2 + (Kn.K^2 n + K^2 n.n + n.Kn)\, n - n.Kn.K^2 n \\ \equiv\; & (n - n)(n - Kn)(n - K^2 n) \equiv 0\,. \end{aligned}$$

B. By Sylvester's formula the expression for any function Fn is

$$\begin{aligned} & \frac{Fg_1.(g_2 - g_3) - Fg_2.(g_1 - g_3) + Fg_3.(g_1 - g_2)}{(g_1 - g_2)(g_1 - g_3)(g_2 - g_3)}\, n^2 \\ & \qquad - \frac{Fg_1.(g_2^2 - g_3^2) - Fg_2.(g_1^2 - g_3^2) + Fg_3.(g_1^2 - g_2^2)}{(g_1 - g_2)(g_1 - g_3)(g_2 - g_3)}\, n \\ & \qquad + \frac{Fg_1.(g_2^2 g_3 - g_2 g_3^2) - Fg_2.(g_1^2 g_3 - g_1 g_3^2) + Fg_3.(g_1^2 g_2 - g_1 g_2^2)}{(g_1 - g_2)(g_1 - g_3)(g_2 - g_3)}\,. \end{aligned}$$

The coefficients of n^2, n, and 1, are symmetric functions of the g's; hence, for the three latent roots, we may substitute n, Kn, and $K^2 n$, when the expression becomes linear in Fn, $F(Kn)$, and $F(K^2 n)$, and, on reducing, the coefficients of $F(Kn)$ and $F(K^2 n)$ will appear as zero, and the coefficients of Fn as

$$\frac{[Kn - K^2 n]\, n^2 - [(Kn)^2 - (K^2 n)^2]\, n + [(Kn)^2.K^2 n - Kn.(K^2 n)^2]}{(n - Kn)(n - K^2 n)(Kn - K^2 n)} = 1\,.$$

C. Since

$$\begin{aligned} & (Fn)^3 - [Fg_1 + Fg_2 + Fg_3](Fn)^2 + [Fg_2.Fg_3 + Fg_3.Fg_1 + Fg_1.Fg_2](Fn) - Fg_1.Fg_2.Fg_3 \\ \equiv\; & (Fn)^3 - [Fn + F(Kn) + F(K^2 n)](Fn)^2 \\ & \qquad + [F(Kn).F(K^2 n) + F(K^2 n).Fn + Fn.F(Kn)](Fn) - Fn.F(Kn).F(K^2 n) \\ \equiv\; & (Fn - Fn)(Fn - F(Kn))(Fn - F(K^2 n)) \equiv 0\,; \end{aligned}$$

by the principle of this section the latent roots of Fn are found among the same functions of the latent roots of n, namely, Fg_1, Fg_2, Fg_3. It might be, however, that these were not all latent roots of Fn; thus, Fg_1 might occur twice, and Fg_2 once as a latent root of Fn, and we might have

$$(Fn - Fg_1)(Fn - Fg_2) = 0\,.$$

In this case the latent roots of $K(Fn) = F(Kn)$ and $K^2(Fn) = F(K^2n)$ would also be Fg_1, occurring twice, and Fg_2: hence

$$Fn + F(Kn) + F(K^2n) = 2Fg_1 + Fg_2.$$

But the left-hand member is a symmetric function of the latent roots of n equal to $Fg_1 + Fg_2 + Fg_3$; hence $Fg_1 = Fg_3$. In like manner, in any other case, we may show that Fg_1, Fg_2, Fg_3, are all latent roots of Fn.

5. The separation of a quaternion q with distinct latent roots into a scalar and a vector part may be arrived at by putting its latent roots equal, respectively, to $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, when

$$q = \varpi \begin{pmatrix} a + b\sqrt{-1} & 0 \\ 0 & a - b\sqrt{-1} \end{pmatrix} \varpi^{-1} = a + b.\varpi \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \varpi^{-1},$$

i. e.,

$$q = a + bi,$$

where i is a non-scalar square root of -1. It should be noted that $T^2q = a^2+b$ is the content of q.

In the same way, if g_1, g_2, g_3, are the three distinct latent roots of n, and

$$\begin{aligned} g_1 &= a + b + c, \\ g_2 &= a + \lambda b + \lambda^2 c, \\ g_3 &= a + \lambda^2 b + \lambda c, \end{aligned}$$

where λ is an imaginary scalar cube root of unity, then

$$\begin{aligned} n &= a.\varpi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \varpi^{-1} + b.\varpi \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} \varpi^{-1} + c.\varpi \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \varpi^{-1} \\ &= a + bi + ci^2, \end{aligned}$$

where i is here a non-scalar cube root of unity.* To select the scalar and non-scalar parts of n we may employ the symbols S and V, as in quaternions;

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* To complete the analogy between nonions and quaternions, the third power of the nonion units should be scalar imaginary cube roots of unity, just as the quaternion units are fourth roots of unity whose squares are scalars (viz. -1); and in the above separation of the nonion n, i should be a cube root of λ. A similar remark applies to the algebras derived from the vids of a matrix of order higher than the third.

and to distinguish the first and second vector parts, we may employ the symbols V_1 and V_2: thus

$$n = Sn + Vn = Sn + V_1 n + V_2 n.\text{*}$$

We then have

$$Kn = Sn + \lambda V_1 n + \lambda^2 V_2 n,$$
$$K^2 n = Sn + \lambda^2 V_1 n + \lambda V_2 n.$$

By definition

$$T^3 n = n.Kn.K^2 n = n.K^2 n.Kn = \text{etc.}\dagger$$
$$= a^3 + b^3 + c^3 - 3abc = S^3 n + T^3 V_1 n + T^3 V_2 n - 3Sn.TV_1 n.TV_2 n.$$

Since, however, $a^3 + b^3 + c^3 - 3abc = g_1 g_2 g_3$, hence $Tn = \sqrt[3]{|n|}$, where $|n|$ denotes the determinant or content of n. Hence it follows that

$$T(nn') = Tn.Tn';$$

and then, if Un denote $n \div Tn$,

$$U(nn') = Un.Un'.$$

6. As an immediate consequence of the separation of a quaternion into a scalar and vector, we have

$$(q - Sq)^2 = V^2 q,$$

which is a form of the identical equation. In matrices of higher orders the identical equation may also be made to appear as an immediate corollary of the separation of a matrix into its ω parts. Thus in the case of a matrix of the third order,

$$(n - a)^3 = (bi + ci^2)^3 = b^3 + c^3 + 3bc\,(bi + ci^2);$$

i. e.,

$$(n - Sn)^3 = T^3 V_1 n + T^3 V_2 n + 3TV_1 n.TV_2 n.(n - Sn).$$

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[*] Neither V_1 nor V_2, as an operator, is distributive over a sum of nonions (as are obviously S and V). I take this occasion to correct an error made in my paper on the theory of matrices in this Journal, Vol. XII, p. 387, where formulae (based on the assumption that V_1 and V_2 are distributive) are given as the nonion analogues of the quaternion formula $V(\alpha\beta + \beta\alpha) = 0$. The true nonion analogue of this formula involves only the symbols S and V. It is given in (8) of this paper.

[†] In quaternions $K(qq') = Kq'.Kq$. This is not a property of the symbol K for matrices of the third and higher orders. If, however, we define a new conjugate as follows:

$$\mathfrak{K}m = Km.K^2 m \ldots . Km^{\omega - 1} \div Tm^{\omega - 2} = Tm.Um^{-1},$$

which also reduces to the quaternion conjugate when $\omega = 2$, then the property in question of the quaternion conjugate is true also of $\mathfrak{K}$; but neither this conjugate nor K is distributive over a sum of matrices, as is the quaternion conjugate over a sum of quaternions.

For a matrix of the fourth order, $m = a + bi + ci^2 + di^3$, where i is now a non-scalar primitive fourth root of unity,

$$\begin{aligned}(m - a)^4 &= (bi + ci^2 + di^3)^4 \\ &= b^4 - c^4 + d^4 + 4bc^2d - 2b^2d^2 \\ &\qquad + 2\,(c^2 + 2bd)(m - a)^2 + 4c\,(b^2 + d^2)(m - a).\end{aligned}$$

In the case of a matrix m of order ω, proceeding in a similar way, the ω^{th} power of the right-hand member (viz. Vm) will be the sum of a scalar $(T^{\omega}Vm)$ and scalar multiples of powers of $Vm = m - Sm$ with exponents from one to $\omega - 2$.

7. On differentiating both members of the identity

$$(q - Sq)^2 = V^2q,$$

we obtain the identical relation between q and any other quaternion $dq = r$, thus:

$$(q - Sq)(r - Sr) + (r - Sr)(q - Sq) = 2SVqVr,$$

i. e., $$qr + rq - 2Sq.r - 2Sr.q + 2SqKr = 0.$$

It is readily seen that the scalar and vector parts of the left-hand member are separately zero.

Since the selective symbols V_1 and V_2 are not distributive, it is not possible to proceed in the same way to obtain the identical relation between two nonions from the identical equation in the form given in the last section; the same is true of matrices of higher orders. We may however dispense with the non-distributive selective symbols V_1 and V_2, etc., and employ only S and V, which are distributive for matrices of any order. Thus for the nonion n we have

$$VV^3n = 3bc\,(bi + ci^2) = \tfrac{3}{2}\,SV^2n.Vn.$$

$$\therefore\ V^3n = SV^3n + VV^3n = SV^3n + \tfrac{3}{2}\,SV^2n.Vn,$$

i. e., $$(n - Sn)^3 = SV^3n + \tfrac{3}{2}\,SV^2n.(n - Sn);$$

and reducing,

$$n^3 - 3Sn.n^2 + 3\,(S^2n - \tfrac{1}{2}\,SV^2n)\,n - (S^3n + SV^3n - \tfrac{3}{2}\,Sn.SV^2n) = 0.$$

If we replace Vn throughout by $n - Sn$, we get the identical equation freed from the symbol V, viz.

$$n^3 - 3Sn.n^2 + \tfrac{3}{2}\,(3S^2n - Sn^2)\,n - (\tfrac{9}{2}\,S^3n - \tfrac{9}{2}\,Sn^2.Sn + Sn^3) = 0.$$

Likewise for the matrix m of the fourth order we have

$$V^4m = SV^4m + VV^4m = SV^4m + \tfrac{4}{3}\, SV^3m \,.\, Vm + 2SV^2m \,.\, V^2m,$$

i. e.,

$$\begin{aligned} m^4 - 4Sm\,.\,m^3 &+ 6\,(S^2m - \tfrac{1}{3}\, SV^2m)\, m^2 \\ &- 4\,(S^3m - SV^2m\,.\,Sm + \tfrac{1}{3}\, SV^3m)\, m \\ &+ (S^4m - 2S^2m\,.\,SV^2m + \tfrac{4}{3}\, Sm\,.\,SV^3m - SV^4m) = 0\,. \end{aligned}$$

Here also we may substitute throughout in the coefficients $m - Sm$ for Vm.

The same process may be extended to matrices of any order.

Given in either form of this section, it is possible, by successive differentiations of the identical equation of a matrix m of order ω, to obtain the $\omega - 1$ identical relations between m and any other matrix m', which, together with the identical equations in m and m', constitute Sylvester's *catena of equations*. Thus in the case of a matrix of order three, differentiating the identical equation in n, we have, if $dn = n'$,

$$\begin{aligned} (n^2n' + nn'n + n'n^2) - 3Sn\,.\,(nn' + n'n) &- 3Sn'\,.\,n^2 \\ &+ 3\,(S^2n - \tfrac{1}{2}\, SV^2n)\, n' + 3\,(2SnSn' - SVnVn')\, n \\ &- (3S^2nSn' + 3S\,.\,V^2nVn' - 3Sn\,.\,SVnVn' - \tfrac{3}{2}\, Sn'\,.\,SV^2n) = 0\,.^* \end{aligned}$$

Freed from the vector symbol, this is

$$\begin{aligned} (n^2n' + nn'n + n'n^2) - 3Sn\,.\,(nn' + n'n) &- 3Sn'\,.\,n^2 \\ &+ \tfrac{3}{2}\,(3S^2n - Sn^2)\, n' + 3\,(3SnSn' - Snn')\, n \\ &- (\tfrac{27}{2}\, S^2n\,.\,Sn' - \tfrac{9}{2}\, Sn^2\,.\,Sn' - 9Sn\,.\,Snn' + 3Sn^2n') = 0\,. \end{aligned}$$

Differentiating again with respect to n, regarding n' as a constant and putting the new $dn = n'$, we obtain a new relation between n and n', which, on dividing through by the factor two, is what would be obtained by substituting in the above n for n' and n' for n. Differentiating once again with the same condition as before, we obtain the identical equation in n', on dividing through by the factor three.

8. In the last section it appeared that the identical equation of n could be obtained, by the proper substitution for Vn and the vector of powers of Vn, from the formula for VV^3n, the nonion analogue of the quaternion formula $VV^2q = 0$.†

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* Since S and V are distributive, just as in quaternions, $dSn = Sdn$, $dVn = Vdn$. Moreover, as in quaternions, $S\,(nn') = Sn\,.\,Sn' + SVnVn'$, while $V\,(nn') = Sn\,.\,Vn' + Sn'\,.\,Vn + V\,.\,VnVn'$; and $SVnVn' = SVn'Vn$, so that a cyclic interchange of a scalar product leaves it unaltered.

† See note on page 165 in reference to the nonion analogue of this formula.

On differentiating both members of the nonion analogue of this formula, we obtain

$$V(V^2n.Vn'+Vn.Vn'.Vn+Vn'.V^2n)=\tfrac{3}{2}SV^2n.Vn'+3SVnVn'.Vn;$$

and differentiating again, regarding n' as a constant,

$$V(Vn.V^2n'+Vn'.Vn.Vn'+V^2n'.Vn)=3SVnVn'.Vn+\tfrac{3}{2}SV^2n'.Vn.$$

These formulae are the nonion analogues of the quaternion formula

$$V(VqVr+VrVq)=0,$$

which can be obtained in like manner from $VV^2q=0$.

Assuming these two formulae, the identical relations between n and n' may be obtained as explicit identities by the proper substitutions in these formulae for Vn, Vn', and the vectors of products of their powers.

If, on the second differentiation of the formula for VV^3n, we put $dn=n''$, still regarding n' as a constant, we shall have

$$V(\Sigma Vn.Vn'.Vn'')=3\,(SVn'Vn''.Vn+SVn''Vn.Vn'+SVnVn'.Vn'')$$

(where $\Sigma Vn.Vn'.Vn''$ denotes the sum of the products of Vn, Vn', Vn'', in all possible orders), which is also an analogue of $V(VqVr+VrVq)=0$. Making the proper substitutions in this formula, we obtain

$$\begin{aligned}&nn'n''+nn''n'+n'nn''+n'n''n+n''nn'+n''n'n\\&-3Sn.(n'n''+n''n')-3Sn'.(n''n+nn'')-3Sn''.(nn'+n'n)\\&+3\,(3SnSn'-Snn')\,n''+3\,(3Sn''Sn-Sn''n)\,n'+3\,(3SnSn'-Snn')\,n''\\&-(27SnSn'Sn''-9SnSn'n''-9Sn'Sn''n-9Sn''Snn'+3Snn'n''+3Snn''n')=0,\end{aligned}$$

which may, of course, also be obtained by differentiating twice the identical equation in n, regarding the first $dn=n'$ as a constant, and putting the second $dn=n''$.

9. The definition of Tn as the cube root of $|n|$ is sufficient to prove that a nonion is separable into the product of a tensor and a versor, since $|nn'|=|n|.|n'|$, and hence the product of two versors is a versor; but, to show the character of the versor part, we may proceed as follows: Since $e^{\theta+\eta}$, $e^{\lambda\theta+\lambda^2\eta}$, $e^{\lambda^2\theta+\lambda\eta}$ may—for proper values of θ and η—have any ratios whatever, we may put

$$n=\varpi\begin{pmatrix}g_1&0&0\\0&g_2&0\\0&0&g_3\end{pmatrix}\varpi^{-1}=\sqrt[3]{g_1g_2g_3}.\varpi\begin{pmatrix}e^{\theta+\eta}&0&0\\0&e^{\lambda\theta+\lambda^2\eta}&0\\0&0&e^{\lambda^2\theta+\lambda\eta}\end{pmatrix}\varpi^{-1}.$$

Denoting $\frac{1}{3}\left[e^{\theta+\eta}+e^{\lambda\theta+\lambda^2\eta}+e^{\lambda^2\theta+\lambda\eta}\right]$, $\frac{1}{3}\left[e^{\theta+\eta}+\lambda^2 e^{\lambda\theta+\lambda^2\eta}+\lambda e^{\lambda^2\theta+\lambda\eta}\right]$,
and $\frac{1}{3}\left[e^{\theta+\eta}+\lambda e^{\lambda\theta+\lambda^2\eta}+\lambda^2 e^{\lambda^2\theta+\lambda\eta}\right]$,

by $f_0(\theta,\eta)$, $f_1(\theta,\eta)$, and $f\ (\theta,\eta)$, respectively, then

$$\begin{aligned} n &= Tn.\varpi\begin{pmatrix} f_0(\theta,\eta)+f_1(\theta,\eta)+f_2(\theta,\eta) & 0 & 0 \\ 0 & f_0(\theta,\eta)+\lambda f_1(\theta,\eta)+\lambda^2 f_2(\theta,\eta) & 0 \\ 0 & 0 & f_0(\theta,\eta)+\lambda^2 f_1(\theta,\eta)+\lambda f_2(\theta,\eta) \end{pmatrix}\varpi^{-1} \\ &= Tn.(f_0(\theta,\eta)+f_1(\theta,\eta).i+f_2(\theta,\eta).i^2), \end{aligned}$$

where $i = V_1 n \div TV_1 n$ is a non-scalar cube root of unity.

If we denote $f_0(\theta,0)=f_0(0,\theta)$, $f_1(\theta,0)=f_2(0,\theta)$, $f_2(\theta,0)=f_1(0,\theta)$ by $f_0\theta, f_1\theta, f_2\theta$, respectively, it is obvious that

$$n = Tn(f_0\theta+f_1\theta.i+f_2\theta.i^2)(f_0\eta+f_2\eta.i+f_1\eta.i^2),^*$$

whence may be derived expressions for the functions $f_0(\theta,\eta)$, etc., in terms of $f_0\theta, f_0\eta$, etc., functions of a single parameter.

Since the latent roots of i and i^2 are 1, λ, λ^2, hence, by Sylvester's formula,

$$\begin{aligned} &e^{\theta i}=f_0\theta+f_1\theta.i+f_2\theta.i^2,\quad e^{\eta i^2}=f_0\eta+f_2\eta.i+f_1\eta.i^2, \\ &\therefore\ n = Tn.e^{\theta i+\eta i^2}. \end{aligned}$$

If, then,

$$\begin{aligned} n' &= Tn'(f_0(\theta',\eta')+f_1(\theta',\eta').i+f_2(\theta',\eta').i^2) \\ &= Tn'(f_0\theta'+f_1\theta'.i+f_2\theta'.i^2)(f_0\eta'+f_2\eta'.i+f_1\eta'.i^2), \\ \therefore\ nn' &= T(nn')(f_0(\theta+\theta',\eta+\eta')+f_1(\theta+\theta',\eta+\eta').i+f_2(\theta+\theta',\eta+\eta').i^2) \\ &= T(nn')(f_0(\theta+\theta')+f_1(\theta+\theta').i+f_2(\theta+\theta').i^2)(f_0(\eta+\eta') \\ &\qquad\qquad +f_2(\eta+\eta').i+f_1(\eta+\eta').i^2). \end{aligned}$$

This formula gives the following formulae for the functions of the sum of two arguments:

$$\begin{aligned} f_0(\theta+\theta',\eta+\eta') &= f_0(\theta,\eta).f_0(\theta',\eta')+f_1(\theta,\eta).f_2(\theta',\eta')+f_2(\theta,\eta).f_1(\theta',\eta'), \\ f_1(\theta+\theta',\eta+\eta') &= f_0(\theta,\eta).f_1(\theta',\eta')+f_1(\theta,\eta).f_0(\theta',\eta')+f_2(\theta,\eta).f_2(\theta',\eta'), \\ f_2(\theta+\theta',\eta+\eta') &= f_0(\theta,\eta).f_2(\theta',\eta')+f_2(\theta,\eta).f_0(\theta',\eta')+f_1(\theta,\eta).f_1(\theta',\eta'), \end{aligned}$$

whence may be derived expressions for $f_0(\theta+\theta')$, etc.

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[*] In the paper referred to in the beginning of this article, in investigating the versor of a nonion, I have considered only $f_0\theta, f_0\eta$, etc., functions of a single parameter; but, inadvertently, the above expression is put equal to $Tn\,(f_0(\theta+\eta)+f_1(\theta+\eta).i+f_2(\theta+\eta).i^2)$, which is false.

From the definition of these functions it is evident that

$$f_0(\lambda\theta) = f_0\theta,\ f_1(\lambda\theta) = \lambda f_0\theta,\ f_2(\lambda\theta) = \lambda^2 f_2\theta,$$

$$f_0(\lambda\theta,\ \lambda^2\eta) = f_0(\theta,\ \eta),\ f_1(\lambda\theta,\ \lambda^2\eta) = \lambda f_1(\theta,\ \eta),\ f_2(\lambda\theta,\ \lambda^2\eta) = \lambda^2 f_2(\theta,\ \eta).$$

$$\begin{aligned}\therefore\ Kn &= Tn\left(f_0(\lambda\theta,\ \lambda^2\eta) + f_1(\lambda\theta,\ \lambda^2\eta).i + f_2(\lambda\theta,\ \lambda^2\eta).i^2\right)\\ &= Tn\left(f_0(\lambda\theta) + f_1(\lambda\theta).i + f_2(\lambda\theta).i^2\right)\left(f_0(\lambda^2\eta) + f_2(\lambda^2\eta).i + f_1(\lambda^2\eta).i^2\right),\\ K^2n &= Tn\left(f_0(\lambda^2\theta,\ \lambda\eta) + f_1(\lambda^2\theta,\ \lambda\eta).i + f_2(\lambda^2\theta,\ \lambda\eta).i^2\right)\\ &= Tn\left(f_0(\lambda^2\theta) + f_1(\lambda^2\theta).i + f_2(\lambda^2\theta).i^2\right)\left(f_0(\lambda\eta) + f_2(\lambda\eta).i + f_1(\lambda\eta).i^2\right).\end{aligned}$$

This result may be obtained more simply from the equation $n = Tn.e^{\theta i + \eta . i^2}$ by means of the formula $(K(Fn) = F(Kn)$.

Since $n^{-1} = (Kn.K^2n) \div T^3n$, hence

$$\begin{aligned}n^{-1} &= (Tn)^{-1}\left(f_0(-\theta,\ -\eta) + f_1(-\theta,\ -\eta).i + f_2(-\theta,\ -\eta).i^2\right)\\ &= (Tn)^{-1}\left(f_0(-\theta) + f_1(-\theta).i + f_2(-\theta).i^2\right)\left(f_0(-\eta) + f_2(-\eta).i + f_1(-\eta).i^2\right).\end{aligned}$$

10. In the Phil. Mag., Nov. 1883, Sylvester has given an application of his formula for any function of a matrix to the problem to find an expression for the p^{th} root of a quaternion; his results involve the trigonometric functions. A like process will give the p^{th} root of a nonion in terms of the functions $f_0(\theta,\ \eta)$, etc.; and a similar remark applies to matrices of any order. But Sylvester's result may be obtained in a simplified form and more easily as follows. If g_1 and g_2 are the distinct latent roots of q, then for a proper value of θ and of the quaternion ϖ, we may put

$$q = Tq.\varpi\begin{pmatrix} e^{\theta\sqrt{-1}} & 0 \\ 0 & e^{\theta\sqrt{-1}} \end{pmatrix}\varpi^{-1} = Tq.\varpi\begin{pmatrix} e^{(\theta + 4k_1\pi)\sqrt{-1}} & 0 \\ 0 & e^{-(\theta - 4k_2\pi)\sqrt{-1}} \end{pmatrix}\varpi^{-1}.$$

$$\begin{aligned}\therefore\ q^{\frac{1}{p}} &= T^{\frac{1}{p}}q.\varpi\begin{pmatrix} e^{\left(\frac{\theta}{p} + \frac{4k_1\pi}{p}\right)\sqrt{-1}} & 0 \\ 0 & e^{-\left(\frac{\theta}{p} - \frac{4k_2\pi}{p}\right)\sqrt{-1}} \end{pmatrix}\varpi^{-1}\\ &= T^{\frac{1}{p}}q.e^{\frac{k_1+k_2}{p}2\pi\sqrt{-1}}.\varpi\begin{pmatrix} e^{\left(\frac{\theta}{p} + 2\frac{k_1 - k_2}{p}\pi\right)\sqrt{-1}} & 0 \\ 0 & e^{-\left(\frac{\theta}{p} + 2\frac{k_1 - k_2}{p}\pi\right)\sqrt{-1}} \end{pmatrix}\varpi^{-1}\\ &= \rho T^{\frac{1}{p}}q.\varpi\begin{pmatrix} \cos\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right) + \sin\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right).\sqrt{-1} & 0 \\ 0 & \cos\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right) - \sin\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right).\sqrt{-1} \end{pmatrix}\varpi^{-1}\\ &= \rho T^{\frac{1}{p}}q\left(\cos\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right) + \sin\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right).UVq\right)\\ &= \rho T^{\frac{1}{p}}q\left(\cos\frac{2k\pi}{p} + \sin\frac{2kp}{p}.UVq\right)\left(\cos\frac{\theta}{p} + \sin\frac{\theta}{p}.UVq\right),\end{aligned}$$

where ρ is a scalar p^{th} root of unity. Whence the p^2 p^{th} roots of q are found among all possible combinations of any p^{th} root of q with the p scalar p^{th} roots of unity and the p quaternion p^{th} roots of unity, whose vector part is parallel to Vq.

From this expression for $q^{\frac{1}{q}}$ we may readily derive

$$q^{\frac{1}{p}} = Aq + B,$$

where

$$A = \rho \frac{T^{\frac{1}{p}}q}{Tq} \frac{\sin\left(\frac{\theta + 2k\pi}{p}\right)}{\sin\theta},$$

$$B = \rho T^{\frac{1}{p}}q \left(\cos\left(\frac{\theta + 2k\pi}{p}\right) - \sin\left(\frac{\theta + 2k\pi}{p}\right)\frac{\cos\theta}{\sin\theta}\right),$$

which are much simpler expressions for A and B than those given by Sylvester in the paper above referred to.

Proceeding in the same way for the nonion n, we have

$$n = Tn.\varpi \begin{pmatrix} e^{\theta + \eta + 6k_1\pi\sqrt{-1}} & 0 & 0 \\ 0 & e^{\lambda\theta + \lambda^2\eta + 6k_2\pi\sqrt{-1}} & 0 \\ 0 & 0 & e^{\lambda^2\theta + \lambda\eta + 6k_3\pi\sqrt{-1}} \end{pmatrix} \varpi^{-1},$$

$$\therefore n^{\frac{1}{p}} = T^{\frac{1}{p}}n.\varpi \begin{pmatrix} e^{\frac{6k_1\pi\sqrt{-1}}{p}} & 0 & 0 \\ 0 & e^{\frac{6k_2\pi\sqrt{-1}}{p}} & 0 \\ 0 & 0 & e^{\frac{6k_3\pi\sqrt{-1}}{p}} \end{pmatrix} \varpi^{-1}.\varpi \begin{pmatrix} e^{\frac{\theta + \eta}{p}} & 0 & 0 \\ 0 & e^{\frac{\lambda\theta + \lambda^2\eta}{p}} & 0 \\ 0 & 0 & e^{\frac{\lambda^2\theta + \lambda\eta}{p}} \end{pmatrix} \varpi^{-1}.$$

If we put

$$\begin{aligned} (\ \ 2k_1 - k_2 - k_3) &= k + k', \\ (-k_1 + 2k_2 - k_3) &= \lambda k + \lambda^2 k', \\ (-k_1 - k_2 + 2k_3) &= \lambda^2 k + \lambda k', \end{aligned}$$

$$\therefore \varpi \begin{pmatrix} e^{\frac{6k_1\pi\sqrt{-1}}{p}} & 0 & 0 \\ 0 & e^{\frac{6k_2\pi\sqrt{-1}}{p}} & 0 \\ 0 & 0 & e^{\frac{6k_3\pi\sqrt{-1}}{p}} \end{pmatrix} \varpi^{-1}$$

$$= e^{\frac{k_1 + k_2 + k_3}{p} 2\pi\sqrt{-1}} \varpi \begin{pmatrix} e^{\frac{k + k'}{p} 2\pi\sqrt{-1}} & 0 & 0 \\ 0 & e^{\frac{\lambda k + \lambda^2 k'}{p} 2\pi\sqrt{-1}} & 0 \\ 0 & 0 & e^{\frac{\lambda^2 k + \lambda k'}{p} 2\pi\sqrt{-1}} \end{pmatrix} \varpi^{-1}.$$

Hence, if we put $2k\pi\sqrt{-1} = \varkappa$ and $2k'\pi\sqrt{-1} = \varkappa'$,

$$\begin{aligned} n^{\frac{1}{p}} &= \rho\,.\,T^{\frac{1}{p}}n\, \Big(f_0\Big(\frac{\varkappa}{p}, \frac{\varkappa'}{p}\Big) + f_1\Big(\frac{\varkappa}{p}, \frac{\varkappa'}{p}\Big).i + f_2\Big(\frac{\varkappa}{p}, \frac{\varkappa'}{p}\Big).i^2 \\ &\qquad\qquad \Big(f_0\Big(\frac{\theta}{p}, \frac{\eta}{p}\Big) + f_1\Big(\frac{\theta}{p}, \frac{\eta}{p}\Big).i + f_2\Big(\frac{\theta}{p}, \frac{\eta}{p}\Big).i^2\Big) \\ &= \rho\,.\,T^{\frac{1}{p}}n\Big(f_0\Big(\frac{\varkappa+\theta}{p}, \frac{\varkappa'+\eta'}{p}\Big) + f_1\Big(\frac{\varkappa+\theta}{p}, \frac{\varkappa'+\eta'}{p}\Big).i + f_2\Big(\frac{\varkappa+\theta}{p}, \frac{\varkappa'+\theta'}{p}\Big).i^2\Big), \end{aligned}$$

where ρ is, as before, a scalar p^{th} root of unity. Consequently the p^3 p^{th} roots of n may all be obtained by combining any p^{th} root of n with the p scalar p^{th} roots of unity and the non-scalar p^{th} roots of unity, whose first and second sub-vector parts are respectively scalar multiples of those of n. From this expression for $n^{\frac{1}{p}}$ it is easy to derive

$$n^{\frac{1}{p}} = An^2 + Bn + C,$$

but the coefficients A, B, C are not very simple expressions.

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